1. Exercises from 4.3

Today we're going to practice some techniques for doing multiple integrals.

PROBLEM 1. *Folland 4.3.1(a)*

-Draw a picture of the region of integration, upper half of the unit disk
-We do the y-integral first, then the x-integral

$$\begin{aligned}\iint_S x+3y^3\,dA &= \int_{-1}^{1}\int_0^{\sqrt{1-x^2}} x+3y^3\,dy\,dx \quad \text{First use linearity to break into 2 integrals}\\ &= \int_{-1}^{1}\int_0^{\sqrt{1-x^2}} x\,dy\,dx + \int_{-1}^{1}\int_0^{\sqrt{1-x^2}} 3y^3\,dy\,dx \quad \text{In first integral, } x \text{ can be considered a constant}\\ &= \int_{-1}^{1} x\sqrt{1-x^2}\,dx + \frac{3}{4}\int_{-1}^{1}\left(1-x^2\right)^2\,dx\end{aligned}$$

- In the first integral, we are integrating an odd function over a symmetric interval; the first integral is zero.
- In the second integral, we can integrate over $[0,1]$ instead and add a factor of two because it is an even function integrated over a symmetric interval.

$$\begin{aligned}\int_{-1}^{1} x\sqrt{1-x^2}\,dx + \frac{3}{4}\int_{-1}^{1}\left(1-x^2\right)^2\,dx &= 0+\frac{3}{2}\int_0^1 (1-x^2)^2\,dx\\ &= \frac{3}{2}\int_0^1 1-2x^2+x^4\,dx\\ &= \frac{3}{2}\left(1-\frac{2}{3}+\frac{1}{5}\right)\\ &= \frac{4}{5}\end{aligned}$$

PROBLEM 2. *Folland 4.3.3(a) Let S be the region bounded by the curves $y=x^3$ and $y=4x$ in the left half of the xy-plane. Write the integral $\int_S f(x,y)\,dx\,dy$ in two different ways.*

The most important part of a problem like this one is to draw a nice picture of the domain of integration.

Integrating over y first:

- The region is bounded from below by $y=4x$
- Bounded from above by $y=x^3$
- The curves intersect at x-values where $x^3-4x=0$, implying $x=0$, or $x=\pm 2$. This gives $x=-2$ in the left half plane.
- We can write the integral then, as:

$$\iint_S f(x,y)\,dA = \int_{-2}^{0}\int_{4x}^{x^3} f(x,y)\,dy\,dx$$

Integrating over x first:

- The region is bounded from the left by $x=y^{1/3}$
- Bounded from the right by $x=\frac{y}{4}$
- The curves intersect at y-values where $y=(-2)^3=-8$ and $y=0$.

- We could have alternatively written the integral as:

$$\iint_S f(x,y)\,dA = \int_{-8}^{0}\int_{y^{1/3}}^{y/4} f(x,y)\,dx\,dy$$

PROBLEM 3. *Folland 4.3.4(c)*

Given an iterated integral, what is the procedure for switching the order of integration? A good strategy is to use the limits of integration to find the region S over which you're integrating, then find the proper way to do the iterated integral in the opposite order.

First task: Find the region S such that

$$\iint_S f(x,y)\,dA = \int_{1}^{2}\int_{0}^{\log x} f(x,y)\,dy\,dx$$

- Notice we integrate over the y-direction first
- The curves which define the limits of y are $y = \log x$ and $y = 0$.
- We integrate from $x = 1$ to $x = 2$ between these curves.
- We can now draw the domain of integration

Task 2: Write out the iterated integral in the opposite order.

- The region is bounded on the left by the curve $x = e^y$
- Bounded on the right by $x = 2$.
- The region is bounded in the y-component by $y = 0$ and $y = \log 2$
- The iterated integral in the opposite order is:

$$\iint_S f(x,y)\,dA = \int_{0}^{\log 2}\int_{e^y}^{2} f(x,y)\,dx\,dy$$

PROBLEM 4. *Folland 4.3.5(b) Calculate*

$$\int_0^1 \int_{\sqrt{x}}^1 \cos(y^3+1)\, dy\, dx$$

We want to switch the order of the integral around. As always, first draw the domain of integration:

So the iterated integral where we integrate over x first has limits $x = 0$ to $x = y^2$, and $y = 0$ to $y = 1$.

$$\begin{aligned}
\iint_S \cos(y^3+1)\, dA &= \int_0^1 \int_0^{y^2} \cos(y^3+1)\, dx\, dy \quad \text{The integrand of the inside integral does not depend on } x \\
&= \int_0^1 \cos(y^3+1) \left[\int_0^{y^2} dx\right] dy \\
&= \int_0^1 y^2 \cos(y^3+1)\, dy \quad \text{Substitute } u = y^3 \\
&= \frac{1}{3}\int_0^1 \cos(u+1)\, du \quad \text{Apply fundamental theorem of calculus} \\
&= \frac{1}{3}(\sin 2 - \sin 1)
\end{aligned}$$